holds:

1) Q is positive definite.

2) Q is positive semidefinite and S_{ep}, T_{ep} are positive definite.

3) Q is positive semidefinite and T_{er} is positive definite and $k_e > 0$.

In order to apply the above procedure to the system defined by Eqs. (5) and (6), we consider the case in which the actuators and sensors are collocated, i.e., $\bar{C} = \bar{B}^T$. Let $P = \text{diag}(P_1, P_2, \ldots, P_n)$, where P_i, $i = 1,2,\ldots,n$ is given by

$$P_i = \begin{bmatrix} 2\alpha\zeta_i\omega_i + \omega_i^2 & \alpha \\ \alpha & 1 \end{bmatrix} \tag{24}$$

Since $\bar{C} = \bar{B}^T$, it follows that $c_i = b_i^T$ and hence Eq. (20) is satisfied.

We now determine conditions under which P is positive definite and Q as defined by Eq. (23) is at least positive semidefinite. Let $\bar{K}_e$ be any positive semidefinite matrix. Then Q can be written as

$$Q = -(PA + A^TP) + 2C^T\bar{K}_eC$$

$$= \text{diag}(Q_1, Q_2, \ldots, Q_n) + 2C^T\bar{K}_eC \tag{25}$$

where Q_i, $i = 1,2,\ldots,n$ is given by

$$Q_i = 2\begin{bmatrix} \alpha\omega_i^2 & 0 \\ 0 & 2\zeta_i\omega_i - \alpha \end{bmatrix} \tag{26}$$

Assuming that $\omega_i > 0$, $i = 1,2,\ldots,n$, we obtain from Eqs. (24) and (26) that P_i is positive definite and Q_i is positive semidefinite if $0 \leq \alpha \leq 2\zeta_i\omega_i$. Hence, P is positive definite and Q is positive semidefinite provided

$$0 \leq \alpha \leq \beta^* \tag{27}$$

where $\beta^* = \underset{i}{\text{Min}}(\beta_i)$ and $\beta_i = 2\zeta_i\omega_i$ is the ith modal damping. Obviously, Q becomes positive definite if strict inequalities hold in Eq. (27), i.e.,

$$0 < \alpha < \beta^* \tag{28}$$

It should be noted that the bound β^* on α is independent of n and requires, by definition, only the knowledge of the lowest modal damping.

For $\zeta_i = \zeta$, a constant, Eq. (27) reduces to

$$0 \leq \alpha \leq 2\zeta\omega^* \tag{29}$$

where $\omega^* = \underset{i}{\text{Min}}(\omega_i)$. From Eq. (29), it is easy to see that the limit on α is less restrictive than that obtained in Ref. 2 where it is shown that stability holds provided $\alpha \leq \text{Min}(\zeta\omega^*, \xi^{-1}\omega^*)$.

Summary and Conclusions

Implicit model reference adaptive control technique provides a promising approach for the control of large structures. Applying this technique to the collocated case, it is shown that the output error approaches zero asymptotically, provided the weighting factor of position to rate measurement is greater than or equal to zero but less than or equal to the lowest modal damping. The control law in the form of integral, proportional, and relay adaptations along with the integral of the output error is proposed.

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Analysis of the Geometric Dilution of Precision Using the Eigenvalue Approach

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I. Introduction

GEOMETRIC dilution of precision (GDOP) has been discussed[1-3] as a criterion for selecting satellites in the global positioning system (GPS). Previous investigations related to GDOP analysis, were performed by Fang,[1] and Brogan,[2] who attempted to develop simple GDOP calculation methods. They employed an eigenvalue approach, since GDOP can be expressed by eigenvalues of the estimation error covariance matrix. However, they did not clarify the relationship between observation directions and eigenvalues. The benefit of the eigenvalue approach is that the relationship between GDOP and observation directions can be expressed explicitly. By making use of this benefit, this Note proposes a new geometrical interpretation of GDOP by using an eigenvalue approach. Based on this interpretation, two simple methods of selecting GPS satellites are used, which do not need matrix inversion calculations.

II. Geometrical Interpretation of Geometric Dilution of Precision

Based on previous discussions, the estimation-error covariance matrix P can be given by $P = (H^TV^{-1}H)^{-1}$, where $H = (A_1, A_2, \ldots, A_n)^T$ and V is a diagonal matrix whose elements are observation error variances. GDOP is defined by the square root of the trace of the estimation error covariance matrix

$$(\text{GDOP})^2 = \text{tr}(H^TV^{-1}H)^{-1} \tag{1}$$

In the GPS navigation problem, $A_i = (a_i, b_i, c_i, 1)$, where (a_i, b_i, c_i) is the line-of-sight vector from a user to a satellite. In the conventional GDOP analysis, it is assumed that if $V = 1$, then, $(\text{GDOP})^2 = \text{tr}(H^TH)^{-1}$. In the case of considering V, if H' is defined by

$$H' = [(1/\sigma_1)A_1, \ldots, (1/\sigma_n)A_n]^T \tag{2}$$

then $(\text{GDOP})^2 = \text{tr}(H'^TH')^{-1}$. Therefore, the characteristics of GDOP depend on the matrix H or the matrix H'. In the

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case of the observation error factor σ_i, the magnitude of the observation vector, that is, the line vector of observation matrix is reduced from $|A_i|$ to $1/\sigma_i\,|A_i|$.

Investigating the characteristics of GDOP by using the eigenvalue approach, first define λ_i $(i=1,...,m)$ to the eigenvalue of H^TH or H'^TH'. Then,

$$\mathrm{tr}(H^TH)=\sum_{i=1}^{m}\lambda_i \qquad \mathrm{tr}(H^TH)^{-1}=\sum_{i=1}^{m}\frac{1}{\lambda_i} \tag{3}$$

Because H^TH is a symmetric matrix, an orthogonal matrix U exists that can diagonalize H^TH as follows:

$$U^TH^THU=D \tag{4}$$

where the diagonal elements of D are eigenvalues of H^TH and other elements of D are zero. The matrix U consists of the eigenvectors $(\xi_1,...,\xi_m)$ of H^TH. From Eq. (4)

$$\lambda_i=\sum_{j=1}^{n}(A_j,\xi_i)^2 \qquad \sum_{j=1}^{n}(A_j,\xi_i)(A_j,\xi_k)=0 \tag{5}$$

That is, the eigenvalues can be represented by the sum of the squares of inner products $(A_j,\ \xi_i)$. Here, assume $\lambda_1\geq...\geq\lambda_m$. From the maximum principle of eigenvalue, ξ_1 maximizes the following function, that is

$$f(\xi)=\sum_{j=1}^{n}(A_j,\xi)^2\rightarrow\max \tag{6}$$

The direction of the vector ξ_1 can be considered to be the most observable direction when observation vectors $A_1,...,A_n$ are given. From the characteristics of eigenvectors, $\xi_2,...,\xi_m$ are perpendicular to ξ_1. Therefore, $\lambda_2,...,\lambda_m$ are determined from the observation information in the $m-1$ dimensional space S^{m-1}, which is perpendicular to the ξ_1 direction. ξ_2 maximizes the function of $f(\xi)$ in the space of S^{m-1}. $(\xi_3,...,\xi_m)$ can be obtained recursively in the same way. The direction of the eigenvector ξ_m corresponding to the minimum eigenvalue, is the least observable direction when observation vectors $A_1,...,A_n$ are given. GDOP can be represented by the sum of the inverse of eigenvalues. Therefore, the minimum eigenvalue has the most significant impact on GDOP.

III. Approximate GDOP Calculation Method and Its Application to GPS Satellite Selection

From the geometrical interpretation of GDOP, the eigenvalues and eigenvectors can be calculated by using the maximization of $f(\xi)$ and the vector orthogonalization. However, the maximization of $f(\xi)$ requires much computation. Therefore, in the proposed approximation method, the eigenvector is given by

$$\xi=\sum_{i=1}^{n}A_i\Big/\Big|\sum_{i=1}^{n}A_i\Big| \tag{7}$$

which maximizes

$$\sum_{i=1}^{n}(A_i,\xi)$$

but does not maximize the function of $f(\xi)$. However, this approximation gives a good approximation in the case of GPS navigation, where $n=m=4$, and is demonstrated through numerical examples in Sec. V. By using the above approximation, GDOP in the GPS navigation can be calculated approximately by the following algorithm.

GDOP Calculation Algorithm

Step 1:

$$\xi_1=\frac{1}{\left|\sum_{i=1}^{4}A_i\right|}\sum_{i=1}^{4}A_i,\quad \lambda_1=\sum_{i=1}^{4}(A_i,\xi_1)^2 \tag{8}$$

Step 2: Assume that $\xi_1,...,\xi_{k-1}$ are given. Then, by using the Gram-Schmidt orthogonalization, each vector's information in the space S^{4-k+1} which is perpendicular to $\xi_1,...,\xi_{k-1}$ is given by

$$A_i^{(k)}=A_i-\sum_{j=1}^{k-1}(A_i,\xi_j)\xi_j \qquad (i=1,...,4) \tag{9}$$

Step 3: Let $A^{*(k)}=\max A_i^{(k)}$. Modify the direction of $A_i^{(k)}$ so that $(A^{*(k)},\ A_i^{(k)})>0$.

Step 4:

$$\xi_k=\frac{1}{\left|\sum_{i=1}^{4}A_i^{(k)}\right|}\sum_{i=1}^{4}A_i^{(k)},\quad \lambda_k=\sum_{i=1}^{4}(A_i,\xi_k)^2 \tag{10}$$

If $k=4$, then go to Step 5, or Step 2.

Step 5: Using the eigenvalues $(\mathrm{GDOP})^2$ can be calculated by

$$\mathrm{GDOP}^2=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4} \tag{11}$$

Here, let's note the relation $\lambda_1+\lambda_2+\lambda_3+\lambda_4=\lambda_T=\mathrm{const}$ in the GPS navigation. Now, define $G(i)$ using $\lambda_1.,,.,\lambda_i$ as follows:

$$G(i)=\frac{1}{\lambda_1}+\cdots\frac{1}{\lambda_i}+\frac{(4-i)^2}{\lambda_T-\sum_{j=1}^{i}\lambda_j} \tag{12}$$

Then, $G(i)\leq(GDOP)^2$. Therefore, in the case of satellite change,[1] where the GDOP G_c of the current satellite selection is given, the satellite selection algorithm can be given by modifying the GDOP calculation algorithm.

Satellite Selection Algorithm

Step 1: same as Step 1 in the GDOP calculation algorithm.

Step 2: if $G(i)\geq G_C$, then do not selet the candidate satellite (stop). If $G(i)<G_C$ and $i<3$, then og to Step 3, or accept the candate (steop).

Step 3: same as steps 2–4 in the GDOP calculation algorithm.

IV. Simple Satellite Selection Method

The satellite selection method developed in Sec. III still requires much computation. In this section, a simpler satellite selection method is presented based on the eigenpolynomial approach. From Eq. (3),

$$\mathrm{GDOP}^2=\sum_{i=1}^{4}\frac{1}{\lambda_i} \tag{13}$$

where the eigenvalue λ_i is the root of $|H^TH-\lambda I|=0$. According to the comment by Kamat[3] from the Cayley-Hamilton theorem,

$$\det(sI-P)=\sum_{i=0}^{4}p_{4-i}s^i \tag{14}$$

Table 1 Example of approximate eigenvalue and eigenvector

	Eigen no.	Eigenvector				Eigenvalue
True calculation	1	−0.3038	−0.0852	−0.4821	0.8173	5.7950
	2	0.9457	−0.1269	−0.0897	0.2854	1.5476
	3	−0.1086	−0.9552	0.2743	0.0218	0.6472
	4	0.0384	−0.2534	−0.8272	−0.5001	0.0103
Approximate calculation	1	−0.2150	−0.0992	−0.4867	0.8409	5.7583
	2	0.9429	−0.2588	−0.0011	0.2099	1.5638
	3	−0.2506	−0.9307	0.2656	−0.0201	0.6674
	4	0.0445	−0.2386	−0.8323	−0.4984	0.0104

Table 2 Approximation error statistics of approximate eigenvalue calculation

Eigenvalue no.	Mean error	Standard deviation
1	0.00967	0.01506
2	0.02338	0.03762
3	0.02317	0.03780
4	0.00613	0.00957
Total experiment number: 5985		

Table 3 Examples of satellite selection simulation

No.	Visible satellites	Selection by true GDOP	Selection by Sec. III	Difference of GDOP^2	Selection by Sec. IV	Difference of GDOP^2
1	2,3,4,10,11,18,19	2,3,4,18	2,3,4,18		2,3,4,18	
2	2,3,4,10,11,18,19	2,3,4,11	2,3,4,11		2,3,4,11	
3	2,3,4,10,11,17,18,19	2,4,11,17	2,10,17,18	0.066	2,4,11,17	
4	2,3,4,10,11,17,18,19	2,10,17,18	2,10,17,18		2,4,11,17	0.214
5	2,3,4,10,11,17,18,19	3,4,11,19	3,4,11,19		4,10,17,19	1.570
6	2,3,4,10,11,17,18,19	3,4,11,19	3,4,11,19		4,10,17,19	0.277
7	2,3,4,10,11,17,18,19	4,10,17,19	3,4,11,19	0.281	4,10,17,19	
8	2,3,10,11,17,18,19	10,17,18,19	10,17,18,19		10,17,18,19	
9	2,3,10,11,17,18,19	3,17,18,19	3,17,18,19		3,17,18,19	
10	2,3,9,10,11,17,18,19	2,9,10,17	2,9,10,17		3,9,17,19	0.139
11	2,3,9,10,11,17,18,19	2,9,10,17	2,9,10,17		3,9,17,19	1.268
12	2,3,9,10,11,17,18,19	3,11,18,19	3,11,18,19		2,9,11,19	1.053
13	2,3,9,10,11,17,18	2,9,10,11	2,9,10,11		2,9,10,11	
14	2,3,9,10,11,17,18	9,10,11,18	9,10,11,18		9,10,11,18	
15	1,2,3,9,10,11,17,18	1,9,11,18	1,2,9,17	0.233	1,9,11,18	
16	1,2,3,9,10,11,17,18	1,2,9,17	1,2,9,17		1,9,11,18	0.401
17	1,2,3,9,10,11,17,18	1,2,9,17	1,2,9,17		1,9,11,18	1.832
18	1,2,3,9,10,11,17,18	3,10,11,18	3,10,11,18		1,3,11,17	0.790
19	1,2,3,9,10,11,17,18	1,3,11,17	3,10,11,18	0.135	1,3,11,17	
20	1,2,3,9,10,17,18	1,2,3,17	1,2,3,17		1,2,3,17	

$$\det[sI-(P)^{-1}] = \sum_{i=0}^{4} \frac{p_i}{p_4} s^i \tag{15}$$

therefore, $(\text{GDOP})^2 = p_3/p_4$. Considering p_3 and p_4, from $p_4 = \det(H^T H) = [\det(H)]^2$

$$\det(H) = \frac{1}{\sigma_1\sigma_2\sigma_3\sigma_4} \begin{vmatrix} a_1-a_4 & b_1-b_4 & c_1-c_4 \\ a_2-a_4 & b_2-b_4 & c_2-c_4 \\ a_3-a_4 & b_3-b_4 & c_3-c_4 \end{vmatrix} \tag{16}$$

If p_3 is computed exactly, then the computed GDOP will agree with Fang's method.[1] But, even if Fang's method is used, the computation is not simple. On the other hand, p_3 has an upper limit. Therefore, the large GDOP value corresponding to the poor geometry seems to be caused by the value of p_4. From this point of view, it seems possible to use p_4 for the measure of satellite selection instead of GDOP, and maximizing p_4 guarantees the nonsingularity in the navigation calculation. Moreover, in the case of satellite switching, where vectors (a_i, b_i, c_i), $i=2,3,4$ are given, the user may check the following simple navigation index for the line-of-sight vector (a_1, b_1, c_1) of the candidate satellite:

$$K = a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 - \beta \tag{17}$$

where

$$\alpha_1 = \begin{vmatrix} b_2-b_4 & c_2-c_4 \\ b_3-b_4 & c_3-c_4 \end{vmatrix} \qquad \alpha_2 = \begin{vmatrix} c_2-c_4 & a_2-a_4 \\ c_3-c_4 & a_3-a_4 \end{vmatrix}$$

$$\alpha_3 = \begin{vmatrix} a_2-a_4 & b_2-b_4 \\ a_3-a_4 & b_3-b_4 \end{vmatrix} \qquad \beta = a_4\alpha_1 + b_4\alpha_2 + c_4\alpha_3$$

V. Numerical Examples

First, the method of GDOP calculation given in Sec. III is examined. Table 1 shows an example of comparison of eigenvalues and eigenvectors between the true GDOP and the approximated GDOP. Table 2 shows the approximation error statistics in a GDOP calculation experiment. These results show that the approximation given in Sec. III seems to be valid. Next, the performance of two satellite selection methods are compared by numerical experiments. A 24-satellite constellation (63-deg orbit inclination, 8 satellites in each orbit) is assumed. Satellite numbers 1–8, 9–16, and 17–24 are in the same orbit planes, respectively. Table 3 shows the satellite selection results by using the proposed methods in Secs. III and IV. The method based on the eigenpolynomial made misselection in several cases. However, the GDOP values in misselection cases are very close to the GDOP of the optimum selection. From these results, the proposed satellite selection methods appear to be effective in the GPS satellite selection problem.

VI. Conclusion

A new geometrical interpretation of the geometric dilution of precision has been proposed, and based on this concept two simple methods of selecting satellites in the global positioning system have been derived which employ the eigenvalue approach and the eigenpolynomial approach. The concept of the geometrical interpretation of the geometric dilution of precision based on the eigenvalue approach seems to be applicable to observation selection in various estimation problems.

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Book Announcements

SZEBEHELY, V., Editor, University of Texas, ***Stability of the Solar System and its Minor Natural and Artificial Bodies,*** D. Reidel, Boston, 1985, 424 pages, $54.00.

Purpose: This bound volume contains the lectures presented at the NATO Advanced Study Institute held in Cortina d'Ampezzo, Italy in August, 1984. The contents listed below represent the major topics discussed at the institute.

Contents: Dynamics of natural and artificial satellites. Theory and application of stability, bifurcation and escape. Resonance and singularities. Hamiltonian mechanics and KAM theory. Chaotic systems and integrability. Geodesic flows, charged particles and extragalactic celestial mechanics.

JUNKINS, J.L., Texas A&M University, and **TURNER, J.D.,** Cambridge Research, ***Optimal Spacecraft Rotational Maneuvers,*** Elsevier, New York, 1986, 516 pages, $109.25.

Purpose: The objective of this text is to provide a unified source of demonstrated methods for computing optimal controls for large-angle nonlinear spacecraft maneuvers.

Contents: Introduction. Geometry and kinematics of rotational motion. Basic principles of dynamics. Rotational dynamics of rigid and multiple rigid body spacecraft. Dynamics of flexible spacecraft. Elements of optimal control theory. Numerical solution of two point boundary value problems. Optimical maneuvers of rigid spacecraft. Optimal large-angle single-axis maneuvers of flexible spacecraft. Frequency-shaped large-angle maneuvers of flexible spacecraft. Computational methods for closed-loop control problems. Appendices. Index.

BIANCHI, G., and **SCHIEHLEN, W.,** Editors, ***Dynamics of Multibody Systems,*** Springer-Verlag, New York, 1986, 323 pages.

Purpose: This book contains the proceedings of the IUTAM/IFToMM Symposium held at Udine, Italy in September, 1985. The contents presented below are the major topics considered at the symposium.

Contents: Computerized formalisms. Modelling techniques. Solution techniques. Flexible systems. Robotics. Gyrodynamics. Dynamics of machines.

HUGHES, P.C., University of Toronto Institute for Aerospace Studies, ***Spacecraft Attitude Dynamics,*** Wiley, New York, 1986, 564 pages, $47.95.

Purpose: This book has been written with students, practicing aerospace engineers and researchers in mind. Vector dynamics and matrix algebra are the only prerequisites. The book contains 250 figures, 175 problems, and 350 references.

Contents: Introduction. Rotational kinematics. Attitude motion equations. Attitude dynamics of a rigid body. Effect of internal energy dissipation on the directional stability of spinning bodies. Directional stability of multispin vehicles. Effect of internal energy dissipation on the directional stability of gyrostats. Spacecraft torques. Gravitational stabilization. Spin stabilization in orbit. Dual-stabilization in orbit: gyrostats and bias momentum satellites. Appendices. References. Index.